

Useful axioms

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Abstract

We give a brief survey on the interplay between forcing axioms and various other non-constructive principles widely used in many fields of abstract mathematics, such as the axiom of choice and Baire’s category theorem.

First of all we outline how, using basic partial order theory, it is possible to reformulate the axiom of choice, Baire’s category theorem, and many large cardinal axioms as specific instances of forcing axioms. We then address forcing axioms with a model-theoretic perspective and outline a deep analogy existing between the standard Łoś Theorem for ultraproducts of first order structures and Shoenfield’s absoluteness for Σ_2^1 -properties. Finally we address the question of whether and to what extent forcing axioms can provide a “complete” semantics for set theory. We argue that to a large extent this is possible for certain initial fragments of the universe of sets: The pioneering work of Woodin on generic absoluteness show that this is the case for the Chang model $L(\text{Ord}^\omega)$ in the presence of large cardinals, and recent works by the author show that this can also be the case for the Chang model $L(\text{Ord}^{\omega_1})$ in the presence of large cardinals and maximal strengthenings of Martin’s maximum or of the proper forcing axiom. The major open question we leave open is whether this situation is peculiar to these Chang models or can be lifted up also to $L(\text{Ord}^\kappa)$ for cardinals $\kappa > \omega_1$.

Introduction

Since its introduction by Cohen in 1963 forcing has been the key and the most effective tool to obtain independence results in set theory. This method has found applications in set theory and in virtually all fields of pure mathematics: in the last forty years natural problems of group theory, functional analysis, operator algebras, general topology, and many other subjects were shown to be undecidable by means of forcing. Starting from the early seventies and during the eighties it became transparent that many of these consistency results could all be derived by a short list of set theoretic principles, which are known in the literature as forcing axioms. These axioms gave set theorists and mathematicians a very powerful tool to obtain independence results: for any given mathematical problem we are most likely able to compute its (possibly different) solutions in the constructible universe L and in models of strong forcing axioms. These axioms settle basic problems in cardinal arithmetic like the size of the continuum and the singular cardinal problem (see among others the works of Foreman, Magidor, Shelah [6], Veličković [21], Todorćević [18], Moore [11], Caicedo and Veličković [3], and the author [22]), as well as combinatorially complicated ones like the basis problem for uncountable linear orders (see Moore’s result [12] which extends previous work of Baumgartner [2], Shelah [16], Todorćević [17], and others). Interesting problems originating from other fields of mathematics and apparently unrelated to set theory have also been settled appealing to forcing axioms, as it is the case (to cite two of the most prominent examples) for Shelah’s results [15] on Whitehead’s problem in group theory and Farah’s result [4] on the non-existence of outer automorphisms of the Calkin algebra in operator algebra. Forcing axioms assert that for a

large class of compact topological spaces X Baire’s category theorem can be strengthened to the statement that any family of \aleph_1 -many dense open subsets of X has non empty intersection. In light of the success these axioms have met in solving problems a convinced platonist may start to argue that these principles may actually give a “complete” theory of a suitable fragment of the universe of sets. However it is not clear how one could formulate such a result. The aim of this paper is to explain in which sense we can show that forcing axioms can give such a “complete” theory and why they are so “useful”.

Section 1 starts showing that two basic non-constructive principles which play a crucial role in the foundations of many mathematical theories, the axiom of choice and Baire’s category theorem, can both be formulated as specific instances of forcing axioms. In section 2 we also argue that many large cardinal axioms can be reformulated in the language of partial orders as specific instances of a more general kind of forcing axioms. Sections 3 and 4 show that Shoenfield’s absoluteness for Σ_2^1 -properties and Loś Theorem for ultraproducts of first order models are two sides of the same coins: recasted in the language of boolean valued models, Shoenfield’s absoluteness shows that there is a more general notion of boolean ultrapower (of which the standard ultrapowers encompassed in Loś Theorem are just special cases) and that in the specific case one takes a boolean ultrapower of a compact, second countable space X , the natural embedding of X in its boolean ultrapower is at least Σ_2 -elementary. Section 5 embarks in a rough analysis of what is a maximal forcing axiom. We are led by two driving observations, one rooted in topological considerations and the other in model-theoretic arguments. First of all we outline how Woodin’s generic absoluteness results for $L(\text{Ord}^\omega)$ entail that in the presence of large cardinals the natural embedding of a separable compact Hausdorff space X in its boolean ultrapowers is not only Σ_2 -elementary but is fully elementary. We then present other recent results by the author which show that, in the presence of natural strengthenings of Martin’s maximum or of the proper forcing axiom, an exact analogue of Woodin’s generic absoluteness result can be established also at the level of the Chang model $L(\text{Ord}^{\omega_1})$. The main open questions left open are whether these generic absoluteness results are specific to the Chang models $L(\text{Ord}^{\omega_i})$ for $i = 0, 1$ or can be replicated also for other cardinals. The paper is meant to be accessible to a wide audience of mathematicians, specifically the first two sections do not require any special familiarity with logic or set theory other than some basic cardinal arithmetic. The third section requires a certain familiarity with first order logic and the basic model theoretic constructions of ultraproducts. The fourth and fifth sections, on the other hand, presume the reader has some familiarity with the forcing method.

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1 The axiom of choice and Baire's category theorem as forcing axioms

The axiom of choice AC and Baire's category theorem BCT are non-constructive principles which play a prominent role in the development of many fields of abstract mathematics. Standard formulations of the axiom of choice and of Baire's category theorem are the following:

Definition 1.1. $\text{AC} \equiv \prod_{i \in I} A_i$ is non-empty for all families of non empty sets $\{A_i : i \in I\}$, i.e. there is a choice function $f : I \rightarrow \bigcup_{i \in I} A_i$ such that $f(i) \in A_i$ for all $i \in I$.

Theorem 1.2. $\text{BCT}_0 \equiv$ For all compact Hausdorff spaces (X, τ) and all countable families $\{A_n : n \in \mathbb{N}\}$ of dense open subsets of X , $\bigcap_{n \in \mathbb{N}} A_n$ is non-empty.

There are large numbers of equivalent formulations of the axiom of choice and it may come as a surprise that one of these is a natural generalization of Baire's category theorem and naturally leads to the notion of forcing axiom.

Definition 1.3. (P, \leq) is a *partial order* if \leq is a reflexive and transitive relation on P .

Notation 1.4. Given a partial order (P, \leq) ,

$$\uparrow A = \{p \in P : \exists q \in A : q \leq p\}$$

denotes the *upward closure* of A and similarly $\downarrow A$ will denote its *downward closure*.

- $A \subseteq P$ is *open* if it is a downward closed subset of P .
- The *order topology* τ_P on P is given by the downward closed subsets of P .
- D is *dense* if for all $p \in P$ there is some $q \in D$ refining p (q refines p if $q \leq p$),
- $G \subseteq P$ is a *filter* if it is upward closed and all $q, p \in G$ have a common refinement $r \in G$.
- p is *incompatible* with q ($p \perp q$) if no $r \in P$ refines both p and q .
- X is a *predense* subset of P if $\downarrow X$ is open dense in P .
- X is an *antichain* of P if it is composed of pairwise incompatible elements, and a maximal one if it is also predense.
- X is a *chain* of P if \leq is a total order on X .

The terminology for open and dense subsets of P comes from the observation that the collection τ_P of downward closed subsets of P is a topology on the space of points P (though in general not a Hausdorff one), whose dense sets are exactly those satisfying the above property. Remark also that the downward closure of a dense set is open dense in this topology.

A simple proof of the Baire Category Theorem is given by a basic enumeration argument:

Lemma 1.5. $\text{BCT}_1 \equiv$ Let (P, \leq) be a partial order and $\{D_n : n \in \mathbb{N}\}$ be a family of predense subsets of P . Then there is a filter $G \subseteq P$ meeting all the sets D_n .

Proof. Build by induction a decreasing chain $\{p_n : n \in \mathbb{N}\}$ with $p_n \in \downarrow D_n$ and $p_{n+1} \leq p_n$ for all n . Let $G = \uparrow \{p_n : n \in \mathbb{N}\}$. Then G meets all the D_n . \square

Baire's category theorem can be proved from the above Lemma (without any use of the axiom of choice) as follows:

Proof of BCT_0 from BCT_1 . Given a compact Hausdorff space (X, τ) and a family of dense open sets $\{D_n : n \in \mathbb{N}\}$ of X , consider the partial order $(\tau \setminus \{\emptyset\}, \subseteq)$ and the family $E_n = \{A \in \tau : \text{Cl}(A) \subseteq D_n\}$. Then it is easily checked that each E_n is dense open in the order topology induced by the partial order $(\tau \setminus \{\emptyset\}, \subseteq)$. By Lemma 1.5, we can find a filter $G \subseteq \tau \setminus \{\emptyset\}$ meeting all the sets E_n . This gives that for all $A_1, \dots, A_n \in G$

$$\text{Cl}(A_1) \cap \dots \cap \text{Cl}(A_n) \supseteq A_1 \cap \dots \cap A_n \supseteq B \neq \emptyset$$

for some $B \in G$ (where $\text{Cl}(A)$ is the closure of $A \subseteq X$ in the topology τ .) By the compactness of (X, τ) ,

$$\bigcap \{\text{Cl}(A) : A \in G\} \neq \emptyset.$$

Any point in this intersection belongs to the intersection of all the open sets D_n . \square

Remark the interplay between the order topology on the partial order $(\tau \setminus \{\emptyset\}, \subseteq)$ and the compact topology τ on X . Modulo the prime ideal theorem (a weak form of the axiom of choice), BCT_1 can also be proved from BCT_0 .

It is less well-known that the axiom of choice has also an equivalent formulation as the existence of filters on posets meeting sufficiently many dense sets. In order to proceed further, we need to introduce the standard notion of forcing axiom.

Definition 1.6. Let κ be a cardinal and (P, \leq) be a partial order.

$\text{FA}_\kappa(P) \equiv$ For all families $\{D_\alpha : \alpha < \kappa\}$ of predense subsets of P , there is a filter G on P meeting all these predense sets.

Given a class Γ of partial orders $\text{FA}_\kappa(\Gamma)$ holds if $\text{FA}_\kappa(P)$ holds for all $P \in \Gamma$.

Definition 1.7. Let λ be a cardinal. A partial order (P, \leq) is $< \lambda$ -closed if every decreasing chain $\{P_\alpha : \alpha < \gamma\}$ indexed by some $\gamma < \lambda$ has a lower bound in P .

Γ_λ denotes the class of $< \lambda$ -closed posets.

It is almost immediate to check that Γ_{\aleph_0} is the class of all posets, and that BCT_1 states that $\text{FA}_{\aleph_0}(\Gamma_{\aleph_0})$ holds. The following formulation of the axiom of choice in terms of forcing axioms has been handed to me by Todorćević, I'm not aware of any published reference. In what follows, let ZF denote the standard first order axiomatization of set theory in the first order language $\{\in, =\}$ (excluding the axiom of choice) and ZFC denote $\text{ZF} +$ the first order formalization of the axiom of choice.

Theorem 1.8. *The axiom of choice AC is equivalent (over the theory ZF) to the assertion that $\text{FA}_\kappa(\Gamma_\kappa)$ holds for all regular cardinals κ .*

We sketch a proof of Theorem 1.8, the interested reader can find a full proof in [13, Chapter 3, Section 2] (see the following hyperlink: *Tesi-Parente*). First of all, it is convenient to prove 1.8 using a different equivalent formulation of the axiom of choice.

Definition 1.9. Let κ be an infinite cardinal. The *principle of dependent choices* DC_κ states the following:

For every non-empty set X and every function $F: X^{<\kappa} \rightarrow \mathcal{P}(X) \setminus \{\emptyset\}$, there exists $g: \kappa \rightarrow X$ such that $g(\alpha) \in F(g \upharpoonright \alpha)$ for all $\alpha < \kappa$.

Lemma 1.10. *AC is equivalent to $\forall \kappa \text{DC}_\kappa$ modulo ZF.*

The reader can find a proof in [13, Theorem 3.2.3]. We prove the Theorem assuming the Lemma:

Proof of Theorem 1.8. We prove by induction on κ that DC_κ is equivalent to $\text{FA}_\kappa(\Gamma_\kappa)$ over the theory $\text{ZF} + \forall \lambda < \kappa \text{DC}_\lambda$. We sketch the ideas for the case κ -regular¹:

Assume DC_κ ; we prove (in ZF) that $\text{FA}_\kappa(\Gamma_\kappa)$ holds. Let (P, \leq) be a $<\kappa$ -closed partially ordered set, and $\{D_\alpha : \alpha < \kappa\} \subseteq \mathcal{P}(P)$ a family of predense subsets of P .

Given a sequence $\langle p_\beta : \beta < \alpha \rangle$ call $\xi_{\vec{p}}$ the least ξ such that $\langle p_\beta : \xi \leq \beta < \alpha \rangle$ is a decreasing chain if such a ξ exists, and fix $\xi_{\vec{p}} = \alpha$ otherwise. Notice that when the length α of \vec{p} is successor then $\xi_{\vec{p}} < \alpha$.

We now define a function $F: P^{<\kappa} \rightarrow \mathcal{P}(P) \setminus \{\emptyset\}$ as follows: given $\alpha < \kappa$ and a sequence $\vec{p} \in P^{<\kappa}$,

$$F(\vec{p}) = \begin{cases} \{p_0\} & \text{if } \xi_{\vec{p}} = \alpha \\ \{d \in \downarrow D_\alpha : d \leq p_\beta \text{ for all } \xi_{\vec{p}} \leq \beta < \alpha\} & \text{otherwise.} \end{cases}$$

The latter set is non-empty since (P, \leq) is $<\kappa$ -closed, $\alpha < \kappa$, and D_α is predense. By DC_κ , we find $g: \kappa \rightarrow P$ such that $g(\alpha) \in F(g \upharpoonright \alpha)$ for all $\alpha < \kappa$. An easy induction shows that for all α the sequence $g \upharpoonright \alpha$ is decreasing, so $g(\alpha) \in \downarrow D_\alpha$ for all $\alpha < \kappa$. Then

$$G = \{p \in P : \text{there exists } \alpha < \kappa \text{ such that } g(\alpha) \leq p\}$$

is a filter on P , such that $G \cap D_\beta \neq \emptyset$ for all $\beta < \kappa$.

Conversely, assume $\text{FA}_\kappa(\Gamma_\kappa)$, we prove (in ZF) that DC_κ holds.

Let X be a non-empty set and $F: X^{<\kappa} \rightarrow \mathcal{P}(X) \setminus \{\emptyset\}$. Define the partially ordered set

$$P = \{s \in X^{<\kappa} : \text{for all } \alpha \in \text{dom}(s), s(\alpha) \in F(s \upharpoonright \alpha)\},$$

with $s \leq t$ if and only if $t \subseteq s$. Let $\lambda < \kappa$ and let $s_0 \geq s_1 \geq \dots \geq s_\alpha \geq \dots$, for $\alpha < \lambda$, be a chain in P . Then $\bigcup_{\alpha < \lambda} s_\alpha$ is clearly a lower bound for the chain. Since κ is regular, we have $\bigcup_{\alpha < \lambda} s_\alpha \in P$ and so P is $<\kappa$ -closed. For every $\alpha < \kappa$, define

$$D_\alpha = \{s \in P : \alpha \in \text{dom}(s)\},$$

and note that D_α is dense in P . Using $\text{FA}_\kappa(\Gamma_\kappa)$, there exists a filter $G \subset P$ such that $G \cap D_\alpha \neq \emptyset$ for all $\alpha < \kappa$. Then $g = \bigcup G$ is a function $g: \kappa \rightarrow X$ such that $g(\alpha) \in F(g \upharpoonright \alpha)$ for all $\alpha < \kappa$. \square

¹In this case the assumption $\text{ZF} + \forall \lambda < \kappa \text{DC}_\lambda$ is not needed, but all the relevant ideas in the proof of the equivalence are already present.

2 Large cardinals as forcing axioms

From now on, we focus on boolean algebras rather than posets.

2.1 A fast briefing on boolean algebras

Definition 2.1. A *boolean algebra* \mathbf{B} is a boolean ring i.e. a ring in which every element is idempotent. Equivalently a boolean algebra is a complemented distributive lattice $(\mathbf{B}, \wedge, \vee, \neg, 0, 1)$ (see [7]).

Notation 2.2. Given a boolean algebra $(\mathbf{B}, \wedge, \vee, \neg, 0, 1)$, the poset $(\mathbf{B}^+; \leq_{\mathbf{B}})$ is given by its non-zero elements, with order relation given by $b \leq_{\mathbf{B}} q$ iff $b \wedge q = b$ iff $b \vee q = q$.

A boolean ring $(\mathbf{B}, +, \cdot, 0, 1)$ has a natural structure of complemented distributive lattice $(\mathbf{B}, \wedge, \vee, \neg, 0, 1)$, for which the sum on the boolean ring becomes the operation Δ of symmetric difference ($a \Delta b = a \vee b \wedge \neg(a \wedge b)$) on the complemented distributive lattice, and the multiplication of the ring the operation \wedge .

We refer to filters, antichains, dense sets, predense sets, open sets on \mathbf{B} , meaning that these notions are declined for the corresponding partial order $(\mathbf{B}^+; \leq_{\mathbf{B}})$.

We also recall the following:

- An *ideal* I on \mathbf{B} is a non-empty downward closed subset of \mathbf{B} with respect to $\leq_{\mathbf{B}}$ which is also closed under \vee . Its *dual filter* \check{I} is the set $\{\neg a : a \in I\}$. It is a filter on the poset $(\mathbf{B}^+; \leq_{\mathbf{B}})$ (equivalently I is an ideal in the boolean ring \mathbf{B}).
- An ideal I on \mathbf{B} is $< \delta$ -complete (δ -complete) if all the subsets of I of size less than δ (of size δ) belong to I .
- A *maximal* ideal I is an ideal properly contained in \mathbf{B} and maximal with respect to this property. Its dual filter is an *ultrafilter*. An ideal I is maximal if and only if $a \in I$ or $\neg a \in I$ for all $a \in \mathbf{B}$.
- \mathbf{B} is $< \delta$ -complete (δ -complete) if all subsets of size less than δ (of size δ) have a supremum and an infimum.
- Given an ideal I on \mathbf{B} , \mathbf{B}/I is the quotient boolean algebra given by equivalence classes $[a]_I$ obtained by $a =_I b$ iff $a \Delta b \in I$.
- \mathbf{B}/I is $< \kappa$ -complete if I is $< \kappa$ -complete.
- \mathbf{B} is *atomless* if there are no minimal elements in the partial order $(\mathbf{B}^+; \leq_{\mathbf{B}})$.
- \mathbf{B} is *atomic* if the set of minimal elements in the partial order $(\mathbf{B}^+; \leq_{\mathbf{B}})$ is open dense.

Usually we insist in the formulation of forcing axioms on the requirement that for certain partial orders P any family of predense subsets of P of some fixed size κ can be met in a single filter. In order to obtain a greater variety of forcing axioms, we need to consider a much richer variety of properties which characterizes the families of predense sets of P which can be met in a single filter. Using boolean algebras, by considering partial orders of the form $(\mathbf{B}^+; \leq_{\mathbf{B}})$ for some boolean algebra \mathbf{B} , we can formulate (using the algebraic structure of \mathbf{B}) a wide spectrum of properties each defining a distinct forcing axiom.

2.2 Measurable cardinals

A cardinal κ is measurable if and only if there is a uniform $< \kappa$ -complete ultrafilter on the boolean algebra $\mathcal{P}(\kappa)$. The requirement that G is uniform amounts to say that G is disjoint from the ideal I on the boolean algebra $(\mathcal{P}(\kappa), \cap, \cup, \emptyset, \kappa)$ given by the bounded subsets of $\mathcal{P}(\kappa)$. This means that we are actually looking for an ultrafilter G on the boolean algebra $\mathcal{P}(\kappa)/I$. This is an atomless boolean algebra which is κ -complete. The requirement that G is $< \kappa$ -complete amounts to ask that G selects an unique member of any partition of κ in $< \kappa$ -many pieces, moreover any maximal antichain $\{[A_i]_I : i < \gamma\}$ in the boolean algebra $\mathcal{P}(\kappa)/I$ of size γ less than κ is induced by a partition of κ in γ -many pairwise disjoint pieces.

All in all, we have the following characterization of measurability:

Definition 2.3. κ is a *measurable* cardinal if and only if there is a ultrafilter G on $\mathcal{P}(\kappa)/I$ (where I is the ideal of bounded subsets of κ) which meets all the maximal antichain on $\mathcal{P}(\kappa)/I$ of size less than κ .

In particular the measurability of κ holds if and only if the partial order $(\mathcal{P}(\kappa)/I)^+$ satisfies a certain forcing axiom stating that certain collections of predense subsets of $(\mathcal{P}(\kappa)/I)^+$ can be simultaneously met in a filter.

We are led to the following definitions:

Definition 2.4. Let (P, \leq) be a partial order and \mathcal{D} be a family of non-empty subsets of P . A filter G on P is \mathcal{D} -generic if $G \cap D$ is non-empty for all $D \in \mathcal{D}$.

Let $\phi(x, y)$ be a property and (P, \leq) a partial order. $\text{FA}_\phi(P)$ holds if for all \mathcal{D} families of predense subsets of P such that $\phi(P, \mathcal{D})$ holds there is some \mathcal{D} -generic filter G on P .

For instance, $\text{FA}_\kappa(P)$ says that $\text{FA}_\phi(P)$ holds for $\phi(x, y)$ being the property:

“ x is a partial order and y is a family of predense subsets of x of size κ ”

The measurability of κ amounts to say that $\text{FA}_\phi(P)$ holds for $\phi(x, y)$ being the property

“ x is the partial order $(\mathcal{P}(\kappa)/I)^+$ and y is the (unique) family of predense subsets of x consisting of maximal antichains of $(\mathcal{P}(\kappa)/I)^+$ of size less than κ ”

We do not want to expand further on this topic but many other large cardinal properties of a cardinal κ can be formulated as axioms of the form $\text{FA}_\phi(P)$ for some property ϕ (for example this is the case for supercompactness, hugeness, almost hugeness, strongness, superstrongness, etc....).

In these first two sections we have already shown that the language of partial orders can accomodate three completely distinct and apparently unrelated families of non-constructive principles which are essential tools in the development of many mathematical theories (as it is the case for the axiom of choice and of Baire’s category theorem) and of crucial importance in the current developments of set theory (as it is the case for large cardinal axioms).

3 Boolean valued models, Łoś theorem, and generic absoluteness

We address here the correlation between forcing axioms and generic absoluteness results. We show how Shoenfield’s absoluteness for Σ_2^1 -properties and Łoś Theorem are two sides of

the same coin: more precisely they are distinct specific cases of a unique general theorem which follows from AC.

After recalling the basic formulation of Łoś Theorem for ultraproducts, we introduce boolean valued models, and we argue that Łoś Theorem for ultraproducts is the specific instance for complete atomic boolean algebras of a more general theorem which applies to a much larger class of boolean valued models. Then we introduce the concept of boolean ultrapower of a first order structure on a Polish space X endowed with Borel predicates R_1, \dots, R_n , and show that Shoenfield's absoluteness for Σ_2^1 -properties amounts to say that the boolean ultrapower of $\langle X, R_1, \dots, R_n \rangle$ by any complete boolean algebra is a Σ_2 -elementary superstructure of $\langle X, R_1, \dots, R_n \rangle$.

3.1 Łoś Theorem

Theorem 3.1. *Let $\{\mathfrak{M}_l : l \in L\}$ be models in a given first order signature*

$$\mathcal{L} = \{R_i : i \in I, f_j : j \in J, c_k : k \in K\},$$

i.e. each $\mathfrak{M}_l = (M_l, R_i^l : i \in I, f_j^l : j \in J, c_k^l : k \in K)$. Let G be a ultrafilter on L (i.e. its dual is a prime ideal on the boolean algebra $\mathcal{P}(L)$). Let

$$[f]_G = \left\{ g \in \prod_{l \in L} M_l : \{l \in L : g(l) = f(l)\} \in G \right\}$$

for each $f \in \prod_{l \in L} M_l$, and set

$$\prod_{l \in L} M_l / G = \left\{ [f]_G : f \in \prod_{l \in L} M_l \right\}.$$

For each $i \in I$ let $\bar{R}_i([f_1]_G, \dots, [f_n]_G)$ hold on $\prod_{l \in L} M_l / G$ if and only if

$$\{l \in L : \mathfrak{M}_l \models R_i^l(f_1(l), \dots, f_n(l))\} \in G.$$

Similarly interpret $\bar{f}_j : \prod_{l \in L} (M_l / G)^n \rightarrow \prod_{l \in L} M_l / G$ and $\bar{c}_k \in \prod_{l \in L} M_l^n / G$ for each $j \in J$ and $k \in K$.

Then:

1. *For all formulae $\phi(x_1, \dots, x_n)$ in the signature \mathcal{L}*

$$\left(\prod_{l \in L} M_l / G, \bar{R}_i : i \in I, \bar{f}_j : j \in J, \bar{c}_k : k \in K \right) \models \phi([f_1]_G, \dots, [f_n]_G)$$

if and only if

$$\{l \in L : \mathfrak{M}_l \models \phi(f_1(l), \dots, f_n(l))\} \in G.$$

2. *Moreover if $\mathfrak{M}_l = \mathfrak{M}$ for all $l \in L$ (i.e. $\prod_{l \in L} M_l / G$ is the ultrapower of M by G), we have that the map $m \mapsto [c_m]_G$ (where $c_m : L \rightarrow M$ is constant with value m) defines an elementary embedding.*

It is a useful exercise to check that the axiom of choice is essentially used in the induction step for existential quantifiers in the proof of Łoś Theorem. Moreover Łoś Theorem is clearly a strenghtnening of the axiom of choice, for the very existence of an element in $\prod_{l \in L} M_l / G$ grants that $\prod_{l \in L} M_l$ is non-empty.

One peculiarity of the above formulation of Łoś theorem is that it applies just to ultrafilters on $\mathcal{P}(X)$. We aim to find a “most” general formulation of this Theorem, which makes sense also for other kind of “ultraproducts” and of ultrafilters on boolean algebras other than $\mathcal{P}(X)$. This forces us to introduce the boolean valued semantics.

3.2 A fast briefing on complete boolean algebras and Stone duality

Recall that for a given topological space (X, τ) the regular open sets are those $A \in \tau$ such that $A = \text{Reg}(A) = \text{Int}(\text{Cl}(A))$ (A coincides with the interior of its closure) and that $\text{RO}(X, \tau)$ is the complete boolean algebra whose elements are regular open sets and whose operations are given by $A \wedge B = A \cap B$, $\bigvee_{i \in I} A_i = \text{Reg}(\bigcup_{i \in I} A_i)$, $\neg A = X \setminus \text{Cl}(A)$.

For any partial order (P, \leq) the map $i : P \rightarrow \text{RO}(P, \tau_P)$ given by $p \mapsto \text{Reg}(\downarrow \{p\})$ is order and incompatibility preserving and embeds P as a dense subset of the non-empty regular open sets in $\text{RO}(P, \tau_P)$.

Recall also that the Stone space $\text{St}(\mathbf{B})$ of a boolean algebra \mathbf{B} is given by its ultrafilters G and it is endowed with a compact topology $\tau_{\mathbf{B}}$ whose clopen sets are the sets $N_b = \{G \in \text{St}(\mathbf{B}) : b \in G\}$ so that the map $b \mapsto N_b$ defines a natural isomorphism of \mathbf{B} with the boolean algebra $\text{CLOP}(\text{St}(\mathbf{B}))$ of clopen subset of $\text{St}(\mathbf{B})$.

Moreover a boolean algebra \mathbf{B} is complete if and only if $\text{CLOP}(\text{St}(\mathbf{B})) = \text{RO}(\text{St}(\mathbf{B}), \tau_{\mathbf{B}})$. Spaces X satisfying the property that its regular open sets are closed are extremally (or extremely) disconnected.

We refer the reader to [7] for a detailed account on these matters.

3.3 Boolean valued models

In a first order model, a formula can be interpreted as true or false. Given a complete boolean algebra \mathbf{B} , \mathbf{B} -boolean valued models generalize Tarski semantics associating to each formula a value in \mathbf{B} , so that propositions are not only true and false anymore (that is, only associated to $1_{\mathbf{B}}$ and $0_{\mathbf{B}}$ respectively), but take also other “intermediate values” of truth. A complete account of the theory of these boolean valued models can be found in [14]. We now recall some basic facts, a detailed account on the material of this section can be found in [20, Chapter 3] (see the following hyperlink: *Tesi-Vaccaro*). In order to avoid unnecessary technicalities, we define boolean valued semantics just for relational first order languages (i.e. signatures with no function symbols).

Definition 3.2. Given a complete boolean algebra \mathbf{B} and a first order relational language

$$\mathcal{L} = \{R_i : i \in I\} \cup \{c_j : j \in J\}$$

a \mathbf{B} -boolean valued model (or \mathbf{B} -valued model) \mathcal{M} in the language \mathcal{L} is a tuple

$$\langle M, =^{\mathcal{M}}, R_i^{\mathcal{M}} : i \in I, c_j^{\mathcal{M}} : j \in J \rangle$$

where:

1. M is a non-empty set, called *domain* of the \mathbf{B} -boolean valued model, whose elements are called \mathbf{B} -names;
2. $=^{\mathcal{M}}$ is the *boolean value* of the equality:

$$\begin{aligned} &=^{\mathcal{M}} : M^2 \rightarrow \mathbf{B} \\ &(\tau, \sigma) \mapsto \llbracket \tau = \sigma \rrbracket_{\mathbf{B}}^{\mathcal{M}} \end{aligned}$$

3. The forcing relation $R_i^{\mathcal{M}}$ is the *boolean interpretation* of the n -ary relation symbol R_i :

$$\begin{aligned} &R_i^{\mathcal{M}} : M^n \rightarrow \mathbf{B} \\ &(\tau_1, \dots, \tau_n) \mapsto \llbracket R_i(\tau_1, \dots, \tau_n) \rrbracket_{\mathbf{B}}^{\mathcal{M}} \end{aligned}$$

4. $c_j^{\mathcal{M}} \in M$ is the *boolean interpretation* of the constant symbol c_j .

We require that the following conditions hold:

- for $\tau, \sigma, \chi \in M$,
 1. $\llbracket \tau = \tau \rrbracket_{\mathbf{B}}^{\mathcal{M}} = 1_{\mathbf{B}}$;
 2. $\llbracket \tau = \sigma \rrbracket_{\mathbf{B}}^{\mathcal{M}} = \llbracket \sigma = \tau \rrbracket_{\mathbf{B}}^{\mathcal{M}}$;
 3. $\llbracket \tau = \sigma \rrbracket_{\mathbf{B}}^{\mathcal{M}} \wedge \llbracket \sigma = \chi \rrbracket_{\mathbf{B}}^{\mathcal{M}} \leq \llbracket \tau = \chi \rrbracket_{\mathbf{B}}^{\mathcal{M}}$;
- for $R \in \mathcal{L}$ with arity n , and $(\tau_1, \dots, \tau_n), (\sigma_1, \dots, \sigma_n) \in M^n$,
 1. $(\bigwedge_{h \in \{1, \dots, n\}} \llbracket \tau_h = \sigma_h \rrbracket_{\mathbf{B}}^{\mathcal{M}}) \wedge \llbracket R(\tau_1, \dots, \tau_n) \rrbracket_{\mathbf{B}}^{\mathcal{M}} \leq \llbracket R(\sigma_1, \dots, \sigma_n) \rrbracket_{\mathbf{B}}^{\mathcal{M}}$;

Given a B-model $\langle M, =^M \rangle$ for equality, a forcing relation R on M is a map $R : M^n \rightarrow \mathbf{B}$ satisfying the above condition for boolean predicates.

The boolean valued semantics is defined as follows:

Definition 3.3. Let

$$\langle M, =^{\mathcal{M}}, R_i^{\mathcal{M}} : i \in I, c_j^{\mathcal{M}} : j \in J \rangle$$

be a B-valued model in a relational language

$$\mathcal{L} = \{R_i : i \in I\} \cup \{c_j : j \in J\},$$

ϕ a \mathcal{L} -formula whose free variables are in $\{x_1, \dots, x_n\}$, and ν a valuation of the free variables in \mathcal{M} whose domain contains $\{x_1, \dots, x_n\}$. Since \mathcal{L} is a relational languages, the terms of a formula are either free variable or constants, let us define $\nu(c_j) = c_j^{\mathcal{M}}$ for c_j a constant of \mathcal{L} . We denote with $\llbracket \phi \rrbracket_{\mathbf{B}}^{\mathcal{M}, \nu}$ the *boolean value* of ϕ with the assignment ν .

Given a formula ϕ , we define recursively $\llbracket \phi \rrbracket_{\mathbf{B}}^{\mathcal{M}, \nu}$ as follows:

- for atomic formulae this is done letting

$$\llbracket t = s \rrbracket_{\mathbf{B}}^{\mathcal{M}, \nu} = \llbracket \nu(t) = \nu(s) \rrbracket_{\mathbf{B}}^{\mathcal{M}},$$

and

$$\llbracket R(t_1, \dots, t_n) \rrbracket_{\mathbf{B}}^{\mathcal{M}, \nu} = \llbracket R(\nu(t_1), \dots, \nu(t_n)) \rrbracket_{\mathbf{B}}^{\mathcal{M}}$$

- if $\phi \equiv \neg \psi$, then

$$\llbracket \phi \rrbracket_{\mathbf{B}}^{\mathcal{M}, \nu} = \neg \llbracket \psi \rrbracket_{\mathbf{B}}^{\mathcal{M}, \nu};$$

- if $\phi \equiv \psi \wedge \theta$, then

$$\llbracket \phi \rrbracket_{\mathbf{B}}^{\mathcal{M}, \nu} = \llbracket \psi \rrbracket_{\mathbf{B}}^{\mathcal{M}, \nu} \wedge \llbracket \theta \rrbracket_{\mathbf{B}}^{\mathcal{M}, \nu};$$

- if $\phi \equiv \exists y \psi(y)$, then

$$\llbracket \phi \rrbracket_{\mathbf{B}}^{\mathcal{M}, \nu} = \bigvee_{\tau \in M} \llbracket \psi(y/\tau) \rrbracket_{\mathbf{B}}^{\mathcal{M}, \nu};$$

If no confusion can arise, we omit the superscripts \mathcal{M}, ν and the subscript \mathbf{B} , and we simply denote the boolean value of a formula ϕ with parameters in \mathcal{M} by $\llbracket \phi \rrbracket$.

With elementary arguments it is possible prove the Soundness Theorem for boolean valued models.

Theorem 3.4 (Soundness Theorem). *Assume \mathcal{L} is a relational language and ϕ is a \mathcal{L} -formula which is syntactically provable by a \mathcal{L} -theory T . Assume each formula in T has boolean value at least $b \in \mathbf{B}$ in a \mathbf{B} -valued model \mathcal{M} with valuation ν . Then $\llbracket \phi \rrbracket_{\mathbf{B}}^{\mathcal{M}, \nu} \geq b$ as well.*

On the other hand the completeness theorem for the boolean valued semantics with respect to first order calculi is a triviality, given that 2 is complete boolean algebra.

We get a standard Tarski model from a \mathbf{B} -valued model passing to a quotient by a ultrafilter $G \subseteq \mathbf{B}$.

Definition 3.5. Take \mathbf{B} a complete boolean algebra, \mathcal{M} a \mathbf{B} -valued model in the language \mathcal{L} , and G a ultrafilter over \mathbf{B} . Consider the following equivalence relation on M :

$$\tau \equiv_G \sigma \Leftrightarrow \llbracket \tau = \sigma \rrbracket \in G$$

The first order model $\mathcal{M}/G = \langle M/G, R_i^{\mathcal{M}/G} : i \in I, c_j^{\mathcal{M}/G} : j \in J \rangle$ is defined letting:

- For any n -ary relation symbol R in \mathcal{L}

$$R^{\mathcal{M}/G} = \{([\tau_1]_G, \dots, [\tau_n]_G) \in (M/G)^n : \llbracket R(\tau_1, \dots, \tau_n) \rrbracket \in G\}.$$

- For any constant symbol c in \mathcal{L}

$$c^{\mathcal{M}/G} = [c^{\mathcal{M}}]_G.$$

If we require \mathcal{M} to satisfy a key additional condition, we get an easy way to control the truth value of formulas in \mathcal{M}/G .

Definition 3.6. A \mathbf{B} -valued model \mathcal{M} for the language \mathcal{L} is *full* if for every \mathcal{L} -formula $\phi(x, \bar{y})$ and every $\bar{\tau} \in M^{|\bar{y}|}$ there is a $\sigma \in M$ such that

$$\llbracket \exists x \phi(x, \bar{\tau}) \rrbracket = \llbracket \phi(\sigma, \bar{\tau}) \rrbracket$$

Theorem 3.7 (Boolean Valued Models Łoś's Theorem). *Assume \mathcal{M} is a full \mathbf{B} -valued model for the relational language \mathcal{L} . Then for every formula $\phi(x_1, \dots, x_n)$ in \mathcal{L} and $(\tau_1, \dots, \tau_n) \in M^n$:*

1. *For all ultrafilters G over \mathbf{B} , $\mathcal{M}/G \models \phi([\tau_1]_G, \dots, [\tau_n]_G)$ if and only if $\llbracket \phi(\tau_1, \dots, \tau_n) \rrbracket \in G$.*
2. *For all $a \in \mathbf{B}$ the following are equivalent:*
 - (a) $\llbracket \phi(f_1, \dots, f_n) \rrbracket \geq a$,
 - (b) $\mathcal{M}/G \models \phi([\tau_1]_G, \dots, [\tau_n]_G)$ for all $G \in N_a$,
 - (c) $\mathcal{M}/G \models \phi([\tau_1]_G, \dots, [\tau_n]_G)$ for densely many $G \in N_a$.

A key observation to relate standard ultraproducts to boolean valued models is the following:

Fact 3.8. *Let $(M_x : x \in X)$ be a family of Tarski-models in the first order relational language \mathcal{L} . Then $N = \prod_{x \in X} M_x$ is a full $\mathcal{P}(X)$ -model letting for each n -ary relation symbol $R \in \mathcal{L}$, $\llbracket R(f_1, \dots, f_n) \rrbracket_{\mathcal{P}(X)} = \{x \in X : M_x \models R(f_1(x), \dots, f_n(x))\}$.*

Let G be any non-principal ultrafilter on X . Then, using the notation of the previous fact, N/G is the familiar ultraproduct of the family $(M_x : x \in X)$ by G , and the usual Łoś Theorem for ultraproducts of Tarski models is the specialization to the case of the full $\mathcal{P}(X)$ -valued model N of Theorem 3.7. Notice that in this special case, if the ultraproduct is an ultrapower of a model M , the embedding $a \mapsto [c_a]_G$ (where $c_a(x) = a$ for all $x \in X$ and $a \in M$) is elementary.

3.4 Boolean ultrapowers of compact Hausdorff spaces and Shoenfield's absoluteness

Take X a set with the discrete topology, and let for any $a \in X$, $G_a \in \text{St}(\mathcal{P}(X))$ denote the principal ultrafilter given by supersets of $\{a\}$. The map $a \mapsto G_a$ embeds X as an open, dense, discrete subspace of $\text{St}(\mathcal{P}(X))$. In particular for any topological space (Y, τ) , any function $f : X \rightarrow Y$ is continuous (since X has the discrete topology) and in the case Y is compact Hausdorff it induces a unique continuous $\bar{f} : \text{St}(\mathcal{P}(X)) \rightarrow Y$ mapping $G \in \text{St}(\mathcal{P}(X))$ to the unique point in Y which is in the intersection of $\{\text{Cl}(A) : A \in \tau, f^{-1}[A] \in G\}$ (we are in the special situation in which $\text{St}(\mathcal{P}(X))$ is also the Stone-Cech compactification of X).

This gives that for any compact Hausdorff space (Y, τ) , the space $C(X, Y) = Y^X$ of (continuous) functions from X to Y is canonically isomorphic to the space $C(\text{St}(\mathcal{P}(X)), Y)$ of continuous functions from $\text{St}(\mathcal{P}(X))$ to Y .

What if we replace $\mathcal{P}(X)$ with an arbitrary (complete) boolean algebra? In view of the above remarks, it is a fair inference to state that $C(\text{St}(\mathbf{B}), Y)$ is the \mathbf{B} -ultrapower of Y for any compact Hausdorff space Y , since this is exactly what occurs for the case $\mathbf{B} = \mathcal{P}(X)$.

Let us examine closely this situation in the case $Y = 2^\omega$ with product topology. This will unfold the relation existing between the notion of Boolean ultrapowers of 2^ω and Shoenfield's absoluteness.

Let us fix \mathbf{B} arbitrary (complete) boolean algebra, and set $M = C(\text{St}(\mathbf{B}), 2^\omega)$. Fix also R a Borel relation on $(2^\omega)^n$. The continuity of an n -tuple $f_1, \dots, f_n \in M$ grants that the set

$$\{G : R(f_1(G), \dots, f_n(G))\} = (f_1 \times \dots \times f_n)^{-1}[R]$$

has the Baire property in $\text{St}(\mathbf{B})$ (i.e. it has symmetric difference with a unique regular open set — see [8, Lemma 11.15, Def. 32.21]), where $f_1 \times \dots \times f_n(G) = (f_1(G), \dots, f_n(G))$. So we can define

$$\begin{aligned} R^M : M^n &\rightarrow \mathbf{B} \\ (f_1, \dots, f_n) &= \text{Reg}(\{G : R(f_1(G), \dots, f_n(G))\}). \end{aligned}$$

Also, since the diagonal is closed in $(2^\omega)^2$,

$$=^M(f, g) = \text{Reg}(\{G : f(G) = g(G)\})$$

is well defined.

It is not hard to check that, for any Borel relation R on $(2^\omega)^n$, the structure $(M, =^M, R^M)$ is a full \mathbf{B} -valued extension of $(2^\omega, =, R)$, where 2^ω is copied inside M as the set of constant functions. It is also not hard to check that whenever G is an ultrafilter on $\text{St}(\mathbf{B})$, the map $i_G : 2^\omega \rightarrow M/G$ given by $x \mapsto [c_x]_G$ (the constant function with value x) defines an injective morphism of the 2-valued structure $(2^\omega, R)$ into the 2-valued structure $(M/G, R^M/G)$. Nonetheless it is not clear whether this morphism is an elementary map or not. This is the case for $\mathbf{B} = \mathcal{P}(X)$, since in this case we are analyzing the standard embedding of the first order structure $(2^\omega, R)$ in its ultrapowers induced by ultrafilters on $\mathcal{P}(X)$. What are the properties of this map if \mathbf{B} is some other complete boolean algebra?

We can relate the degree of elementarity of the map i_G with Shoenfield's absoluteness for Σ_2^1 -properties. This can be done if one is eager to accept as a black-box the identification of the \mathbf{B} -valued model $C(\text{St}(\mathbf{B}), 2^\omega)$ with the \mathbf{B} -valued model given by the family of \mathbf{B} -names for elements of 2^ω in $V^{\mathbf{B}}$ (which is the canonical \mathbf{B} -valued model for set theory), we will expand further on this identification in the next section. Modulo this identity,

Shoenfield's absoluteness can be recasted as a statement about boolean valued models. We choose to name Cohen's absoluteness the following statement, which gives (as we will see) an equivalent reformulation of Shoenfield's absoluteness. Its proof (as we will see in the next section) ultimately relies on Cohen's forcing theorem, hence the name.

Theorem 3.9 (Cohen's absoluteness). *Assume \mathbf{B} is a complete boolean algebra and $R \subseteq (2^\omega)^n$ is a Borel relation. Let $M = C(\text{St}(\mathbf{B}), 2^\omega)$ and $G \in \text{St}(\mathbf{B})$. Then*

$$(2^\omega, =, R) \prec_{\Sigma_2} (M/G, =^M /G, R^M /G).$$

4 Getting Cohen's absoluteness from Baire's category Theorem

Let us now show how Theorem 3.9 is once again a consequence of forcing axioms. To do so, we dwelve deeper into set theoretic techniques and assume the reader has some acquaintance with the forcing method. We give below a brief recall sufficient for our aims.

4.1 Forcing

Let V denote the standard universe of sets and ZFC the standard first order axiomatization of set theory by the Zermelo-Frankel axioms. For any complete boolean algebra $\mathbf{B} \in V$ let

$$V^{\mathbf{B}} = \{f : V^{\mathbf{B}} \rightarrow \mathbf{B}\}$$

be the class of \mathbf{B} -names with boolean relations $\in^{\mathbf{B}}, \subseteq^{\mathbf{B}}, =^{\mathbf{B}} : (V^{\mathbf{B}})^2 \rightarrow \mathbf{B}$ given by:

1.

$$\in^{\mathbf{B}}(\tau, \sigma) = \llbracket \tau \in \sigma \rrbracket = \bigvee_{\tau_0 \in \text{dom}(\sigma)} (\llbracket \tau = \tau_0 \rrbracket \wedge \sigma(\tau_0)).$$

2.

$$\subseteq^{\mathbf{B}}(\tau, \sigma) = \llbracket \tau \subseteq \sigma \rrbracket = \bigwedge_{\sigma_0 \in \text{dom}(\tau)} (\tau(\sigma_0) \rightarrow \llbracket \sigma_0 \in \sigma \rrbracket) = \bigwedge_{\sigma_0 \in \text{dom}(\tau)} (\neg \tau(\sigma_0) \vee \llbracket \sigma_0 \in \sigma \rrbracket).$$

3.

$$=^{\mathbf{B}}(\tau, \sigma) = \llbracket \tau = \sigma \rrbracket = \llbracket \tau \subseteq \sigma \rrbracket \wedge \llbracket \sigma \subseteq \tau \rrbracket.$$

Theorem 4.1 (Cohen's forcing theorem I). *$(V^{\mathbf{B}}, \in^{\mathbf{B}}, =^{\mathbf{B}})$ is a full boolean valued model which assigns the boolean value $1_{\mathbf{B}}$ to all axioms $\phi \in \text{ZFC}$.*

V is copied inside $V^{\mathbf{B}}$ as the family of \mathbf{B} -names $\check{a} = \{\langle \check{b}, 1_{\mathbf{B}} \rangle : b \in a\}$ and has the property that for all Σ_0 -formulae (i.e with quantifiers bounded to range over sets) $\phi(x_0, \dots, x_n)$ and $a_0, \dots, a_n \in V$

$$\llbracket \phi(\check{a}_0, \dots, \check{a}_n) \rrbracket = 1_{\mathbf{B}} \text{ if and only if } V \models \phi(a_0, \dots, a_n).$$

This procedure can be formalized in any first order model $(M, E, =)$ of ZFC for any $\mathbf{B} \in M$ such that $(M, E, =)$ models that \mathbf{B} is a complete boolean algebra.

Two ingredients are still missing to prove Cohen's absoluteness (Theorem 3.9) from Baire's category theorem: the notion of M -generic filter and the duality between $C(\text{St}(\mathbf{B}), 2^\omega)$ and the \mathbf{B} -names in $V^{\mathbf{B}}$ for elements of 2^ω . We first deal with the duality.

4.2 $C(\text{St}(\mathbf{B}), 2^\omega)$ is the family of \mathbf{B} -names for elements of 2^ω

Definition 4.2. Let \mathbf{B} be a complete boolean algebra. Let $\sigma \in V^\mathbf{B}$ be a \mathbf{B} -name such that $\llbracket \sigma : \check{\omega} \rightarrow \check{2} \rrbracket_\mathbf{B} = 1_\mathbf{B}$. We define $f_\sigma : \text{St}(\mathbf{B}) \rightarrow 2^\omega$ by

$$f_\sigma(G)(n) = i \iff \llbracket \sigma(\check{n}) = \check{i} \rrbracket \in G.$$

Conversely assume $g : \text{St}(\mathbf{B}) \rightarrow 2^\omega$ is a continuous function, then define

$$\tau_g = \{ \langle \check{n}, i \rangle, \{ G : g(G)(n) = i \} : n \in \omega, i < 2 \} \in V^\mathbf{B}.$$

Observe indeed that

$$\{ G \in \text{St}(\mathbf{B}) : g(G)(n) = i \} = g^{-1}[N_{n,i}],$$

where $N_{n,i} = \{ f \in 2^\omega : f(n) = i \}$. Since g is continuous, $g^{-1}[N_{n,i}]$ is clopen and so it is an element of \mathbf{B} .

We can prove the following duality:

Proposition 4.3. Assume that $\llbracket \sigma : \check{\omega} \rightarrow \check{2} \rrbracket_\mathbf{B} = 1_\mathbf{B}$ and $g : \text{St}(\mathbf{B}) \rightarrow 2^\omega$ is continuous. Then

1. $\tau_g \in V^\mathbf{B}$;
2. $f_\sigma : \text{St}(\mathbf{B}) \rightarrow 2^\omega$ is continuous;
3. $\llbracket \tau_{f_\sigma} = \sigma \rrbracket_\mathbf{B} = 1_\mathbf{B}$;
4. $f_{\tau_g} = g$.

In particular letting

$$(2^\omega)^\mathbf{B} = \left\{ \sigma \in V^\mathbf{B} : \llbracket \sigma : \check{\omega} \rightarrow \check{2} \rrbracket_\mathbf{B} = 1_\mathbf{B} \right\},$$

the 2-valued models $((2^\omega)^\mathbf{B}/G, =^\mathbf{B}/G)$ and $(C(\text{St}(\mathbf{B}), 2^\omega), =^{\text{St}(\mathbf{B})}/G)$ are isomorphic for all $G \in \text{St}(\mathbf{B})$ via the map $[g]_G \mapsto [\tau_g]_G$.

This is just part of the duality, as the duality can lift the isomorphism also to all \mathbf{B} -Baire relations on 2^ω , among which are all Borel relations. Recall that for any given topological space (X, τ) a subset Y of X is meager for τ if Y is contained in the countable union of closed nowhere dense (i.e. with complement dense open) subsets of X . Y has the Baire property if $Y \Delta A$ is meager for some unique regular open set $A \in \tau$.

Definition 4.4. $R \subseteq (2^\omega)^n$ is a \mathbf{B} -Baire subset of $(2^\omega)^n$ if for all continuous functions $f_1, \dots, f_n : \text{St}(\mathbf{B}) \rightarrow 2^\omega$ we have that

$$(f_1 \times \dots \times f_n)^{-1}[A] = \{ G : f_1 \times \dots \times f_n(G) \in A \}$$

has the Baire property in $\text{St}(\mathbf{B})$.

$R \subseteq (2^\omega)^n$ is universally Baire if it is \mathbf{B} -Baire for all complete boolean algebras \mathbf{B} .

It can be shown in ZFC that Borel (and even analytic) subsets of $(2^\omega)^n$ are universally Baire (see [8, Def. 32.21]).

An important result of Feng, Magidor, and Woodin [5] can be restated as follows:

Theorem 4.5. $R \subseteq (2^\omega)^n$ is \mathbf{B} -Baire if and only if there exist $\dot{R}^{\mathbf{B}} \in V^{\mathbf{B}}$ such that

$$\llbracket \dot{R}^{\mathbf{B}} \subseteq (2^\omega)^n \rrbracket = 1_{\mathbf{B}},$$

and for all $\tau_1, \dots, \tau_n \in (2^\omega)^{\mathbf{B}}$

$$\text{Reg}(\{G : R(f_{\tau_1}(G), \dots, f_{\tau_n}(G))\}) = \llbracket (\tau_1, \dots, \tau_n) \in \dot{R}^{\mathbf{B}} \rrbracket.$$

In particular an easy Corollary is the following:

Theorem 4.6. Let $R \subseteq (2^\omega)^n$ be a \mathbf{B} -baire relation. Then the map $[f]_G \mapsto [\tau_f]_G$ implements an isomorphism between

$$\langle C(\text{St}(\mathbf{B})/G, R^{\text{St}(\mathbf{B})}/G) \rangle \cong \langle (2^\omega)^{\mathbf{B}}/G, \dot{R}^{\mathbf{B}}/G \rangle$$

for any $G \in \text{St}(\mathbf{B})$.

These results can be suitably generalized to arbitrary Polish spaces. We refer the reader to [19] and [20].

4.3 M -generic filters and Cohen's absoluteness

Definition 4.7. Let (P, \leq) be a partial order and M be a set. A subset G of P is M -generic if $G \cap D$ is non-empty for all $D \in M$ predense subset of P .

By BCT_1 every countable set M admits M -generic filters for all partial orders P .

Theorem 4.8 (Cohen's forcing theorem II). Assume (N, \in) is a transitive model of ZFC, $\mathbf{B} \in N$ is a complete boolean algebra in N , and $G \in \text{St}(\mathbf{B})$ is an N -generic filter for \mathbf{B}^+ .

Let

$$\begin{aligned} \text{val}_G : N^{\mathbf{B}} &\rightarrow V \\ \sigma &\mapsto \sigma_G = \{ \tau_G : \exists b \in G \langle \tau, b \rangle \in \sigma \}, \end{aligned}$$

and $N[G] = \text{val}_G[N^{\mathbf{B}}]$.

Then $N[G]$ is transitive, and the map $[\sigma]_G \mapsto \sigma_G$ is the Mostowski collapse of the Tarski models $\langle N^{\mathbf{B}}/G, \in^{\mathbf{B}}/G \rangle$ and induces an isomorphism of this model with the model $\langle N[G], \in \rangle$.

In particular for all formulae $\phi(x_1, \dots, x_n)$ and $\tau_1, \dots, \tau_n \in N^{\mathbf{B}}$

$$\langle N[G], \in \rangle \models \phi((\tau_1)_G, \dots, (\tau_n)_G)$$

if and only if $\llbracket \phi(\tau_1, \dots, \tau_n) \rrbracket \in G$.

Recall that:

- For any infinite cardinal λ , H_λ is the set of all sets $a \in V$ such that $|\text{trcl}(a)| < \lambda$ (where $\text{trcl}(a)$ is the transitive closure of the set a).
- If κ is a strongly inaccessible cardinal (i.e. regular and strong limit), H_κ is a transitive model of ZFC.
- A property $R \subseteq (2^\omega)^n$ is Σ_2^1 , if it is of the form

$$R = \{ (a_1, \dots, a_n) \in (2^\omega)^n : \exists y \in 2^\omega \forall x \in 2^\omega S(x, y, a_1, \dots, a_n) \}$$

with $S \subseteq (2^\omega)^{n+2}$ a Borel relation.

- If $\phi(x_0, \dots, x_n)$ is a Σ_0 -formula and $M \subseteq N$ are transitive sets or classes, then for all $a_0, \dots, a_n \in M$

$$M \models \phi(a_0, \dots, a_n) \text{ if and only if } N \models \phi(a_0, \dots, a_n).$$

Observe that for any theory $T \supseteq \text{ZFC}$ there is a recursive translation of Σ_2^1 -properties (provably Σ_2^1 over T) into Σ_1 -properties over H_{ω_1} (provably Σ_1 over the same theory T) [8, Lemma 25.25].

Lemma 4.9. *Assume $\phi(x, r)$ is a Σ_0 -formula in the parameter $\vec{r} \in (2^\omega)^n$. Then the following are equivalent:*

1. $H_{\omega_1} \models \exists x \phi(x, r)$.
2. For all complete boolean algebra \mathbb{B} $\llbracket \exists x \phi(x, r) \rrbracket = 1_{\mathbb{B}}$.
3. There is a complete boolean algebra \mathbb{B} such that $\llbracket \exists x \phi(x, r) \rrbracket > 0_{\mathbb{B}}$.

Summing up we get: a Σ_2^1 -statement holds in V iff the corresponding Σ_1 -statement over H_{ω_1} holds in some model of the form $V^{\mathbb{B}}/G$.

Combining the above Lemma with Proposition 4.3, we can easily infer the proof of Theorem 3.9.

Proof. We shall actually prove the following slightly stronger formulation of the non-trivial direction in the three equivalences above:

$$H_{\omega_1} \models \exists x \phi(x, r) \text{ if } \llbracket \exists x \phi(x, r) \rrbracket > 0_{\mathbb{B}} \text{ for some complete boolean algebra } \mathbb{B} \in V.$$

To simplify the exposition we prove it with the further assumption that there exists an inaccessible cardinal $\kappa > \mathbb{B}$. With the obvious care in details the large cardinal assumption can be removed. So assume $\phi(x, \vec{y})$ is a Σ_0 -formula and $\llbracket \exists x \phi(x, \vec{r}) \rrbracket > 0_{\mathbb{B}}$ for some complete boolean algebra $\mathbb{B} \in V$ with parameters $\vec{r} \in (2^\omega)^n$. Pick a model $M \in V$ such that $M \prec (H_\kappa)^V$, M is countable in V , and $\mathbb{B}, \vec{r} \in M$. Let $\pi_M : M \rightarrow N$ be its transitive collapse (i.e. $\pi_M(a) = \pi_M[a \cap M]$ for all $a \in M$) and $\mathbb{Q} = \pi_M(\mathbb{B})$. Notice also that $\pi_M(\vec{r}) = \vec{r}$: since $\omega \in M$ is a definable ordinal contained in M , $\pi_M(\omega) = \pi_M[\omega] = \omega$, consequently π_M fixes also all the elements in $2^\omega \cap M$.

Since π_M is an isomorphism of M with N ,

$$N \models \text{ZFC} \wedge (b = \llbracket \exists x \phi(x, \vec{r}) \rrbracket > 0_{\mathbb{Q}}).$$

Now let $G \in V$ be N -generic for \mathbb{Q} with $b \in G$ (G exists since N is countable), then, by Cohen's theorem of forcing applied in V to N , we have that $N[G] \models \exists x \phi(x, \vec{r})$. So we can pick $a \in N[G]$ such that $N[G] \models \phi(a, \vec{r})$. Since $N, G \in (H_{\omega_1})^V$, we have that V models that $N[G] \in H_{\omega_1}^V$ and thus V models that a as well belongs to $H_{\omega_1}^V$. Since $\phi(x, \vec{y})$ is a Σ_0 -formula, V models that $\phi(a, \vec{r})$ is absolute between the transitive sets $N[G] \subset H_{\omega_1}$ to which a, \vec{r} belong. In particular a witnesses in V that $H_{\omega_1}^V \models \exists x \phi(x, \vec{r})$. \square

5 Maximal forcing axioms

Guided by all the previous results we want to formulate maximal forcing axioms. We pursue two directions:

1. A direction led by topological considerations: we have seen that $\text{FA}_{\aleph_0}(P)$ holds for any partial order P , and that AC is equivalent to the satisfaction of $\text{FA}_\lambda(P)$ for all regular λ and all $< \lambda$ -closed posets P .

We want to isolate the largest possible class of partial orders Γ_λ for which $\text{FA}_\lambda(P)$ holds for all $P \in \Gamma_\lambda$. The case $\lambda = \aleph_0$ is handled by Baire's category theorem, that shows that Γ_{\aleph_0} is the class of all posets. The case $\lambda = \aleph_1$ is settled by the work of Foreman, Magidor, and Shelah and leads to Martin's maximum. On the other hand, the case $\lambda > \aleph_1$ is wide open and until recently only partial results have been obtained. New techniques to handle the case $\lambda = \aleph_2$ are being developed (notably by Neeman, and with important contributions by Veličković, Cox, Krueger, and others), however the full import of their possible applications is not clear yet.

2. A direction led by model-theoretic considerations: Baire's category theorem implies that the natural embedding of 2^ω into $C(\text{St}(\mathbf{B}), 2^\omega)/G$ is Σ_2 -elementary, whenever 2^ω is endowed with \mathbf{B} -baire predicates (among which all the Borel predicates). We want to reinforce this theorem in two directions:

- (A) We want to be able to infer that (at least for Borel predicates) the natural embedding of 2^ω into $C(\text{St}(\mathbf{B}), 2^\omega)/G$ yields a full elementary embedding of 2^ω into $C(\text{St}(\mathbf{B}), 2^\omega)/G$.
- (B) We want to be able to define boolean ultrapowers $M^{\mathbf{B}}$ also for other first order structures M other than 2^ω and be able to infer that the natural embedding of M into $M^{\mathbf{B}}/G$ is elementary for these boolean ultrapowers.

Both directions (the topological and the model-theoretic) converge towards the isolation of certain natural forcing axioms. Moreover for each cardinal λ , the relevant structures for which we can define a natural notion of boolean ultrapower are either the structure H_{λ^+} , or the Chang model $L(\text{Ord}^\lambda)$.

We have a satisfactory understanding of the maximal forcing axioms one can get following both directions for the cases $\lambda = \aleph_0, \aleph_1$. The main open question remains how to isolate (if at all possible) the maximal forcing axioms for $\lambda > \aleph_1$.

5.1 Woodin's generic absoluteness

We start by the model-theoretic direction, following Woodin's work in Ω -logic. Observe that a set theorist works either with first order calculus to justify some proofs over ZFC , or with forcing to obtain independence results over ZFC . However, in axiom systems extending ZFC there seems to be a gap between what we can achieve by ordinary proofs and the independence results that we can obtain by means of forcing. To close this gap we miss two desirable features of a "complete" first order theory T that contains ZFC , specifically with respect to the semantics given by the class of boolean valued models of T :

- T is complete with respect to its intended semantics, i.e for all statements ϕ only one among $T + \phi$ and $T + \neg\phi$ is forceable.
- Forceability over T should correspond to a notion of derivability with respect to some proof system, for instance derivability with respect to a standard first order calculus for T .

Both statements appear to be rather bold and have to be handled with care: Consider for example the statement $\omega = \omega_1$ in a theory T extending ZFC with the statements ω is

the first infinite cardinal and ω_1 is the first uncountable cardinal. Then clearly T proves $|\omega| \neq |\omega_1|$, while if one forces with $\text{Coll}(\omega, \omega_1)$ one produce a model of set theory where this equality holds (however the formula ω_1 is the first uncountable cardinal is now false in this model).

At first glance, this suggests that as we expand the language for T , forcing starts to act randomly on the formulae of T , switching the truth value of its formulae with parameters in ways which it does not seem simple to describe. However the above difficulties are raised essentially by our lack of attention to define the type of formulae for which we aim to have the completeness of T with respect to forceability. We can show that when the formulae are prescribed to talk only about a suitable initial segment of the set theoretic universe (i.e. H_{ω_1} or $L(\text{Ord}^\omega)$), and we consider only forcings that preserve the intended meaning of the parameters by which we enriched the language of T (i.e. parameters in H_{ω_1}), this random behaviour of forcing does not show up anymore.

We take a platonist stance towards set theory, thus we have one canonical model V of ZFC of which we try to uncover the truths. To do this, we may use model theoretic techniques that produce new models of the part of $\text{Th}(V)$ on which we are confident. This certainly includes ZFC, and (if we are platonists) all the axioms of large cardinals.

We may start our quest for uncovering the truth in V by first settling the theory of $H_{\omega_1}^V$ (the hereditarily countable sets), then the theory of $H_{\omega_2}^V$ (the sets of hereditarily cardinality \aleph_1) and so on and so forth, thus covering step by step all infinite cardinals. To proceed we need some definitions:

Definition 5.1. Given a theory $T \supseteq \text{ZFC}$ and a family Γ of partial orders definable in T , we say that ϕ is Γ -consistent for T if T proves that there exists a complete boolean algebra $\mathbb{B} \in \Gamma$ such that $\llbracket \phi \rrbracket_{\mathbb{B}} > 0_{\mathbb{B}}$.

Given a model V of ZFC we say that V models that ϕ is Γ -consistent if ϕ is Γ -consistent for $\text{Th}(V)$.

Definition 5.2. Let

$$T \supseteq \text{ZFC} + \{\lambda \text{ is an infinite cardinal}\}$$

Ω_λ is the definable (in T) class of partial orders P which satisfy $\text{FA}_\lambda(P)$.

In particular Baire's category theorem amounts to say that Ω_{\aleph_0} is the class of all partial orders (denoted by Woodin as the class Ω). The following is a careful reformulation of Lemma 4.9 which do not require any commitment on the ontology of V .

Lemma 5.3 (Cohen's absoluteness Lemma). *Assume $T \supseteq \text{ZFC} + \{p \subseteq \omega\}$ and $\phi(x, p)$ is a Σ_0 -formula. Then the following are equivalent:*

- $T \vdash \exists x \phi(x, p)$,
- $T \vdash \exists x \phi(x, p)$ is Ω -consistent.

This shows that for Σ_1 -formulae with real parameters the desired overlap between the ordinary notion of provability and the semantic notion of forceability is provable in ZFC. Now it is natural to ask if we can expand the above in at least two directions:

1. Increase the complexity of the formula,
2. Increase the language allowing parameters also for other infinite cardinals.

The second direction will be pursued in the next subsection. Concerning the first direction, the extent by which we can increase the complexity of the formula requires once again some attention to the semantical interpretation of its parameters and its quantifiers. We have already observed that the formula $\omega = \omega_1$ is inconsistent but Ω -consistent in a language with parameters for ω and ω_1 . One of Woodin's main achievements² in Ω -logic show that if we restrict the semantic interpretation of ϕ to range over the structure $L([\text{Ord}]^{\aleph_0})$ and we assume large cardinal axioms, we can get a full correctness and completeness result [9, Corollary 3.1.7]:

Theorem 5.4 (Woodin). *Assume*

$T \supseteq \text{ZFC} + \{p \subset \omega\} + \text{there are class many Woodin cardinals which are limits of Woodin cardinals},$

$\phi(x, y)$ is any formula in free variables x, y , $A \subseteq (2^\omega)^n$ is universally Baire. Then the following are equivalent (where \dot{A}^B is the B -name given by Theorem 4.5 lifting A to V^B):

- $T \vdash [L([\text{Ord}]^{\aleph_0}, A) \models \phi(p, A)],$
- $T \vdash \exists B \left\| L([\text{Ord}]^{\aleph_0}, \dot{A}^B) \models \phi(p, \dot{A}^B) \right\| > 0_B,$
- $T \vdash \forall B \left\| L([\text{Ord}]^{\aleph_0}, \dot{A}^B) \models \phi(p, \dot{A}^B) \right\| = 1_B.$

Remark that since $H_{\omega_1} \subseteq L([\text{Ord}]^{\aleph_0})$, via Theorem 4.5 and natural generalizations of [8, Lemma 25.25] establishing a correspondence between Σ_{n+1}^1 -properties and Σ_n -properties over H_{ω_1} , we obtain that for any complete boolean algebra B and any Σ_n^1 -predicate $R \subseteq (2^\omega)^n$ the map $x \mapsto [c_x]_G$ of $(2^\omega, R)$ into $(C(\text{St}(B, 2^\omega), R^{\text{St}(B)}))$ is an elementary embedding. In particular the above theorem provide a first fully satisfactory answer to the question of whether the natural embedding of 2^ω in its boolean ultrapowers can be elementary: the answer is yes if we accept the existence of large cardinal axioms!

The natural question to address now is whether we can step up this result also for uncountable λ . If so in which form?

5.2 Martin's maximum MM

Let us now address the quest for maximal forcing axioms from the topological direction. Specifically: what is the largest class of partial orders Γ for which we can predicate $\text{FA}_{\aleph_1}(\Gamma)$?

Shelah proved that $\text{FA}_{\aleph_1}(P)$ fails for any P which does not preserve stationary subsets of ω_1 . Nonetheless it cannot be decided in ZFC whether this is a necessary condition for a poset P in order to have the failure of $\text{FA}_{\aleph_1}(P)$. For example let P be a forcing which shoots a club of ordertype ω_1 through a projectively stationary and costationary subset of $P_{\omega_1}(\omega_2)$ by selecting countable initial segments of this club: It is provable in ZFC that P preserve stationary subsets of ω_1 for all such P . However in L , $\text{FA}_{\aleph_1}(P)$ fails for some such P while in a model of Martin's maximum MM, $\text{FA}_{\aleph_1}(P)$ holds for all such P .

The remarkable result of Foreman, Magidor, and Shelah [6] is that the above necessary condition is consistently also a sufficient condition: it can be forced that $\text{FA}_{\aleph_1}(P)$ holds if and only if P is a forcing notion preserving all stationary subsets of ω_1 . This axiom is known in the literature as Martin's maximum MM. In particular MM realizes a maximality property for forcing axioms.

²We follow Larson's presentation as in [9].

5.3 Boosting Woodin's absoluteness to $L(\text{Ord}^{\omega_1})$: MM^{+++}

Let us now come back to the model theoretic approach and focus on whether and how MM and its variants can control the first order theory with parameters in H_{ω_2} of the structure H_{ω_2} or more generally of the Chang model $L([\text{Ord}]^{\leq \aleph_1})$. We want to address the question of whether MM (which can be seen as a maximal topological strengthening at the level of \aleph_1 of Baire's category theorem) can give also a form of model theoretic maximality for $L([\text{Ord}]^{\leq \aleph_1})$ analogous to the one provided by Woodin's absoluteness results for $L([\text{Ord}]^{\aleph_0})$.

A natural approach to study the Chang model $L([\text{Ord}]^{\leq \aleph_1})$ is to expand the language of ZFC to include constants for all elements of H_{ω_2} and the basic relations between these elements:

Definition 5.5. Let V be a model of ZFC and $\lambda \in V$ be a cardinal. The Σ_0 -diagram of H_λ^V is given by the theory

$$\{\phi(p) : p \in H_\lambda^V, \phi(p) \text{ a } \Sigma_0\text{-formula true in } V\}.$$

Following our approach, as we already know that $\text{ZFC} + \text{large cardinal axioms}$ settles the theory of $L([\text{Ord}]^{\aleph_0})$ with parameters in H_{ω_1} , the natural theory of V that we should look at is:

$$T = \text{ZFC} + \text{large cardinal axioms} + \Sigma_0\text{-diagram of } H_{\omega_2}.$$

Now consider any model M of T , obtained using model-theoretic techniques.

Assume in particular that M is a “monster model”: it contains V , and for some completion \bar{T} of T , M is a model of \bar{T} that amalgamates “all” possible models of \bar{T} and realizes all consistent types of \bar{T} with parameters in $H_{\omega_2}^V$. If there is any hope that \bar{T} is really the theory of V we are aiming for, we should at least require that $V \prec_{\Sigma_1} M$.

Once we make this requirement we notice the following:

$$V \cap \mathbf{NS}_{\omega_1}^M = \mathbf{NS}_{\omega_1}^V,$$

where \mathbf{NS}_{ω_1} is the ideal of non-stationary subsets of ω_1 . If this was not the case, then for some S stationary and costationary in V , M models that S is not stationary, i.e. that there is a club of ω_1 disjoint from S . Since $V \prec_{\Sigma_1} M$ such a club can be found in V . This means that V already models that S is non stationary. Now the formula $S \cap C = \emptyset$ and C is a club is Σ_0 and thus it is part of the Σ_0 -diagram of $H_{\omega_2}^V$. However this contradicts the assumption that S is stationary and costationary in V which is expressed by the fact that the above formula is not part of the Σ_0 -elementary diagram of V . This shows that $V \not\prec_{\Sigma_1} M$.

Thus any “monster” model M as above should be correct about the non-stationary ideal, so we better add this ideal as a predicate to the Σ_0 -diagram of $H_{\omega_2}^V$, to rule out models of the completions of T which cannot even be Σ_1 -superstructures of V . Remark that on the forcing side, this is immediately leading to the notion of stationary set preserving forcing: if we want to use forcing to produce such approximations of “monster” models while preserving the fact of being a Σ_1 -elementary superstructure of V with respect to T , we have to restrict our attention to stationary set preserving forcings. Let us denote by SSP the class of stationary set preserving posets and we obtain that Martin's maximum asserts that $\text{FA}_{\aleph_1}(P)$ holds for all SSP -partial orders P .

A final remark concerning the construction of this “monster” model is given by a modular reformulation of Cohen's absoluteness (see for a proof [24, Lemma 1.3] and recall that for each cardinal λ , Ω_λ is the class of partial orders P satisfying $\text{FA}_\lambda(P)$):

Lemma 5.6 (Generalized Cohen’s absoluteness Lemma). *Assume*

$$T \supseteq \text{ZFC} + \{p \subset \lambda\} + \{\lambda \text{ is an infinite cardinal}\}$$

and $\phi(x, p)$ is a Σ_0 -formula. Then the following are equivalent:

- $T \vdash \exists x \phi(x, p)$,
- $T \vdash \exists x \phi(x, p)$ is Ω_λ -consistent.

In particular the assertion that $\text{FA}_{\aleph_1}(\text{SSP})$ holds entails that the above Lemma at the level of \aleph_1 holds for the largest possible class of complete boolean algebras.

Subject to the limitations we have outlined the best possible result we can hope for is to find a theory

$$T_1 \supset T$$

such that:

1. T_1 proves some natural strengthening of Martin’s maximum,
2. For any $T_2 \supseteq T_1$ and any formula $\phi(S)$ relativized to the Chang model $L([\text{Ord}]^{\leq \aleph_1})$ with parameter $S \subset \omega_1$ and a predicate for the non-stationary ideal NS_{ω_1} , $\phi(S)$ is provable in T_2 if and only if the theory $T_1 + \phi(S)$ is Ω_{\aleph_1} -consistent for T_2 .

We shall separately give arguments to justify these two requirements.

First of all, Martin’s maximum can be seen as a natural topological formulation of a strengthening of the axiom of choice, since countably closed forcings are stationary set preserving and the axiom of choice implies that $\text{FA}_{\aleph_1}(P)$ holds for all countably closed forcings.

On the other hand, it is also well known by means of Stone duality that for any compact Hausdorff topological space (X, τ) , the intersection of a family of λ -many dense open sets is non-empty if and only if the partial order $P = (\tau \setminus \{\emptyset\}, \supseteq)$ is such that $\text{FA}_\lambda(P)$ holds.

Martin’s maximum is a maximal topological strengthening of Baire’s category theorem, asserting that the intersection of a family of \aleph_1 -many dense open sets of a compact Hausdorff space is non-empty, whenever this assertion is not outward contradictory. Denying Martin’s maximum is not required by the known constraints we have to impose on T in order to get a complete extension of T .

The second requirement (2) above is the best possible form of completeness theorem we can currently formulate: there may be interesting model theoretic tools to produce models of T which are not encompassed by forcing, however we haven’t as yet developed powerful techniques to exploit them in the study of models of ZFC. Moreover the second requirement (2) shows that forcing becomes a powerful proof tool in the presence of strong forcing axioms, since it transforms a validity problem in a consistency problem (or equivalently, consistency results in proofs).

The main result regarding strengthenings of Martin’s maximum yielding analogues of Woodin’s absoluteness results is the following [23].

Theorem 5.7. *Let ZFC^* stands for³*

$$\text{ZFC} + \text{there are class many } \Sigma_2\text{-reflecting cardinals}$$

³ δ is a Σ_2 -reflecting cardinal if it is inaccessible and for all formulae ϕ and $A \in V_\delta$, there exists α such that $V_\alpha \models \phi(A)$ if and only if there exists an $\alpha < \delta$ with this property.

and T^* be any theory extending

$$\text{ZFC}^* + \text{MM}^{+++} + \omega_1 \text{ is the first uncountable cardinal} + S \subset \omega_1.$$

Then for any formula⁴ $\phi(S)$ the following are equivalent:

1. $T^* \vdash [L([\text{Ord}]^{\leq \aleph_1}) \models \phi(S)]$,
2. $T^* \vdash \text{MM}^{+++}$ and $[L([\text{Ord}]^{\leq \aleph_1}) \models \phi(S)]$ are jointly Ω_{\aleph_1} -consistent.

It can be seen that the result is sharp: the work of Asperó [1] and Larson [10] shows that we cannot obtain the above completeness and correctness result relative to forcing axioms which are just slightly weaker than MM^{++} (which is a natural strengthening of MM). It remains open whether our axiom MM^{+++} is really stronger than MM^{++} in the presence of large cardinals.

On the other hand a feature of this result which is worth exploring is that it is modular: There is a finite explicit list of properties such that for any class Γ of posets satisfying them, one can prove the consistency of an axiom $\text{CFA}(\Gamma)$ fixing the theory of the Chang model $L(\text{Ord}^\kappa)$ with respect to all forcing in Γ preserving $\text{CFA}(\Gamma)$ (where κ (provided it exists) is the largest regular cardinal such that $\text{FA}_\kappa(P)$ holds for all $P \in \Gamma$). $\text{CFA}(\text{SSP})$ is the forcing axiom MM^{+++} of the above theorem, however Asperó has shown that the classes Γ of proper, semiproper, ω^ω -bounding, preserving a Suslin tree, and combinations and variants thereof are all such that $\text{CFA}(\Gamma)$ is consistent. It is an interesting question to understand whether there can be classes Γ such that $\text{CFA}(\Gamma)$ is consistent and $\text{FA}_\kappa(P)$ holds for all $P \in \Gamma$ for some regular $\kappa > \omega_1$.

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⁴If we allow formulae of arbitrary complexity, we do not need to enrich the language with a predicate for the non-stationary ideal, since this ideal is a definable predicate over H_{ω_2} (though defined by a Σ_1 -property) and thus can be incorporated as a part of the formula.

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